Group-strategyproof cost-sharing mechanisms

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Introduction

- A set of agents $A = \{1, \ldots, n\}$
- A facility that can serve multiple agents
- Cost of serving $S \subseteq A$ is $c(S)$
- Valuation v_i for each agent
- Valuations are private knowledge

Introduction

- Each agent announces b_i (possibly $b_i \neq v_i$)
- **Mechanism**: Decide
  (i) set $O(b)$ to be serviced
  (ii) payment $p_i(b)$ for each agent
- Utility of agent i:

$$v_i \cdot a_i(b) - p_i(b),$$

where $a_i(b) = 1$, iff $i \in O(b)$ and $a_i(b) = 0$ otherwise

Axioms of Cost-sharing

Volunetary Participation (VP):
Non-negative utilities

Non Positive Transfer (NPT):
Non-negative payments

Consumer Sovereignty (CS):
Guaranteed service for any agent i given high b_i
(more than some $b_i^* \in \mathbb{R}$)

Group-strategyproofness (GSP):
No coalition can benefit by misreporting

Cost Sharing Schemes

Definition

A cost sharing scheme is a function $\xi : A \times 2^A \to \mathbb{R}^+ \cup \{0\}$, s.t. $\xi(i, S) > 0 \Rightarrow i \in S$.

Definition

A cost sharing scheme ξ is α-budget balanced for c iff for all $S \subseteq A$, $\alpha c(S) \leq \sum_{i \in S} \xi(i, S) \leq c(S)$.

Definition

A cost sharing scheme ξ is cross monotone iff for all $S, T \subseteq A$ and $i \in S$: $\xi(i, S) \geq \xi(i, S \cup T)$.

Moulin Mechanism

Algorithm 1 Moulin Mechanism

Require: cross monotone ξ, vector b

$S \leftarrow A$

repeat

$S \leftarrow \{i \in S \mid b_i \geq \xi(i, S)\}$

until $\forall i \in S,\ b_i \geq \xi(i, S))\}$

Service S and charge each agent $\xi(i, S)$

Theorem ([1])

Moulin Mechanism is GSP.

Proof Sketch

Definition

Given a bid vector b a set S is feasible iff $\forall i \in S$, $b_i \geq \xi(i, S)$.

Lemma

If ξ is cross monotone, then there is a **unique maximal** *feasible set. This set is also the outcome of Moulin mechanism.*

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- All feasible sets are subsets of the current outcome

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- A successful coalition targets feasible outcomes
- All feasible sets are subsets of the current outcome
- No utility gain!

Submodular Cost

Definition

A cost function c is submodular iff for all $S, T \subseteq A$:

$$c(S \cup T) + c(S \cap T) \leq c(S) + c(T).$$

Theorem ([2])

For every submodular cost function c there is a 1-budget balanced cross monotone cost sharing scheme.

Limitations of cross monotonicity

Problem	Lower bound	
Edge-cover	$\frac{1}{2}$	[3]
Set-cover	$\frac{2}{\sqrt{n}}$	[3]
Vertex-cover	$\frac{2}{n^{1/3}}$	[3]
Facility Location	$\frac{1}{3}$	[3]
Steiner Forest	$\frac{1}{2}$	[4]

Beyond Cross monotonicity

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$\{1\}/\{2\}$	10	10

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Algorithm 6 GSP mechanism for ξ

$S \leftarrow \emptyset$
if $b_1 > \xi(1, S \cup \{1\})$ **then**
  $S \leftarrow S \cup \{1\}$
end if
if $b_2 \geq \xi(2, S \cup \{2\})$ **then**
  $S \leftarrow S \cup \{2\}\}$
end if

Partial Characterization

Theorem ([3])

Every GSP mechanism defines a cost sharing scheme

Definition

A cost sharing scheme ξ is semi-cross monotone iff for all $S \subseteq A$ and $i \in S$, either

$$\forall j \in S \setminus \{i\}\colon \xi(j, S) \geq \xi(j, S \setminus \{i\}) \text{ or}$$

$$\forall j \in S \setminus \{i\}\colon \xi(j, S) \leq \xi(j, S \setminus \{i\}).$$

Theorem ([3])

The cost-sharing schemes that give rise to GSP mechanisms are semi-cross monotone.

Proof Sketch

1. $\xi(j, S \setminus \{i\}) > \xi(j, S)$ and $\xi(k, S \setminus \{i\}) < \xi(k, S)$

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2. Two possible outcomes $S \setminus \{i\}$ and S with player i indifferent

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3. Player i can bid high to help k

Proof Sketch

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2. Two possible outcomes $S \setminus \{i\}$ and S with player i indifferent
3. Player i can bid high to help k
4. Player i can bid low to help j

Proof Sketch

1. $\xi(j, S \setminus \{i\}) > \xi(j, S)$ and $\xi(k, S \setminus \{i\}) < \xi(k, S)$
2. Two possible outcomes $S \setminus \{i\}$ and S with player i indifferent
3. Player i can bid high to help k
4. Player i can bid low to help j
5. Always a coalition!

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No GSP mechanism for ξ

Challenge: Find a complete characterization of GSP cost sharing schemes

Fence Monotonicity

Definition

Given $L, U \subseteq A$ and $i \in U$ let

$$\xi^*(i, L, U) = \min_{L \cup \{i\} \subseteq S \subseteq U} \xi(i, S).$$

Remark

For ξ cross monotone: $\xi^*(i, L, U) = \xi(i, U)$.

Fence Monotonicity

First Condition

There must be some S, where $L \subseteq S \subseteq U$ such that for all $i \in S$, $\xi(i, S) = \xi^*(i, L, U)$.

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- Cross monotone $\Rightarrow$ First Condition

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- For cross monotone ξ, set $S = U$.
- Cross monotone $\Rightarrow$ First Condition
- Equivalent with semi-cross monotonicity for $|U \setminus L| = 1$
- Fence monotone $\Rightarrow$ Semi-cross monotone

Fence Monotonicity

Second Condition

For every $i \in U \setminus L$, there must be some S_i, where $L \subseteq S_i \subseteq U$ and $i \in S_i$ such that for all $i \in S_i \setminus L$, $\xi(i, S) = \xi^*(i, L, U)$.

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Second Condition

For every $i \in U \setminus L$, there must be some S_i, where $L \subseteq S_i \subseteq U$ and $i \in S_i$ such that for all $i \in S_i \setminus L$, $\xi(i, S) = \xi^*(i, L, U)$.

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Fence Monotonicity

Second Condition

For every $i \in U \setminus L$, there must be some S_i, where $L \subseteq S_i \subseteq U$ and $i \in S_i$ such that for all $i \in S_i \setminus L$, $\xi(i, S) = \xi^*(i, L, U)$.

- For cross monotone ξ, set $S_i = U$.
- Cross monotone $\Rightarrow$ Second Condition
- For $|U \setminus L| = 2$: if $\xi(i, S \setminus \{j\}) < \xi(i, S)$ then $\xi(j, S \setminus \{i\}) \leq \xi(i, S)$

Fence Monotonicity

Third Condition

If for $C \subset U$, there is $j \in C$ with $\xi(j, C) < \xi^*(j, L, U)$ then $\exists T \subseteq L \setminus C$ s.t. $\forall i \in T$, $\xi(i, C \cup T) = \xi^*(i, L, U)$.

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- Cross monotone: if condition always false
- Cross monotone $\Rightarrow$ Third Condition
- For $|U \setminus L| = 1$
  1. if $\xi(i, S \setminus \{j\}) < \xi(i, S)$:
     $\xi(j, S \setminus \{i\}) \geq \xi(i, S) \Rightarrow \xi(j, S \setminus \{i\}) = \xi(i, S)$
  2. if $\xi(i, S \setminus \{j\}) < \xi(i, S)$ and $\xi(k, S \setminus \{i\}) < \xi(k, S)$:
     $\xi(k, S \setminus \{j\}) \leq \xi(i, S \setminus \{i\})$

Necessity

Theorem ([5])

Every cost sharing scheme gives rise to GSP mechanism is fence monotone.

Proof Sketch.

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- Define harm relation iff $\xi(i, L, U) > \xi(i, L \cup \{j\}, U)$

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- Induction on $|U \setminus L|$; Base trivially satisfied
- Define harm relation iff $\xi(i, L, U) > \xi(i, L \cup \{j\}, U)$
- Induction hypothesis (Property 3) $\Rightarrow$ harm relation strict partial order

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- **Important**: existence of sinks!

Proof Sketch (cont.)

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  - Ind. hyp. cond 1. and GSP $\Rightarrow$ Alloc. prop. A

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  - Ind. hyp. cond 1. and GSP $\Rightarrow$ Alloc. prop. A
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- **Second Condition**

Proof Sketch (cont.)

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  - Alloc. prop. $A \Rightarrow$ Ind. St. Cond. 1
- **Second Condition**
  - Ind. hyp. Cond. 1 $\Rightarrow$ Ind. St. cond. 2 (sinks)

Proof Sketch (cont.)

- **First Condition**
  - Ind. hyp. cond 1. and GSP $\Rightarrow$ Alloc. prop. A
  - Alloc. prop. $A \Rightarrow$ Ind. St. Cond. 1
- **Second Condition**
  - Ind. hyp. Cond. 1 $\Rightarrow$ Ind. St. cond. 2 (sinks)
  - Ind. hyp. Cond. 2 and GSP $\Rightarrow$ Ind. St. Cond. 2 (rest)

Proof Sketch (cont.)

- **First Condition**
  - Ind. hyp. cond 1. and GSP $\Rightarrow$ Alloc. prop. A
  - Alloc. prop. $A \Rightarrow$ Ind. St. Cond. 1
- **Second Condition**
  - Ind. hyp. Cond. 1 $\Rightarrow$ Ind. St. cond. 2 (sinks)
  - Ind. hyp. Cond. 2 and GSP $\Rightarrow$ Ind. St. Cond. 2 (rest)
- **Third Condition**

Proof Sketch (cont.)

- **First Condition**
  - Ind. hyp. cond 1. and GSP $\Rightarrow$ Alloc. prop. A
  - Alloc. prop. $A \Rightarrow$ Ind. St. Cond. 1
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- **Third Condition**
  - Alloc. prop. A and GSP $\Rightarrow$ Alloc. prop. C

Proof Sketch (cont.)

- **First Condition**
  - Ind. hyp. cond 1. and GSP $\Rightarrow$ Alloc. prop. A
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  - Ind. hyp. Cond. 2 and GSP $\Rightarrow$ Ind. St. Cond. 2 (rest)
- **Third Condition**
  - Alloc. prop. A and GSP $\Rightarrow$ Alloc. prop. C
  - Alloc. prop. C $\Rightarrow$ Ind. Step cond. 2

Stable pairs

Definition

L, U are stable at b, iff

1. $\forall i \in L,\ b_i > \xi^*(i, L, U)$
2. $\forall i \in U \setminus L,\ b_i = \xi^*(i, L, U)$
3. $\forall R \subseteq A \setminus U,\ \exists i \in R\colon\ b_i < \xi^*(i, L, U \cup R)$

Definition

Given two sets L, U and a bid vector b a valid tie breaking rule $\sigma(L, U, b) = S \in A$ satisfies: $\xi(i, S) = \xi^*(i, L, U)$, for all $i \in S$.

Fencing mechanisms

Algorithm 7 Fencing mechanism

Require: Fence monotone ξ, valid tie-breaking rule σ for ξ, and bid vector b

Find stable pair L, U

$S \leftarrow \sigma(L, U, b)$

return S

Theorem

Fencing mechanisms are GSP.

Yielding Process

Algorithm 8 Yield Process

Require: ξ, vector b, set L

$U \leftarrow A$

repeat

$U \leftarrow L \cup \{i \in U \setminus L \mid b_i \geq \xi^*(i, L, U)\}$

until $\forall i \in U \setminus L,\ b_i \geq \xi^*(i, L, U)\}$

return U

Lemma

If L, U is stable at b, then L yields U.

Proof Sketch

Lemma

Given a fence monotone ξ there is at most one stable pair at each bid vector

Lemma

Given a fence monotone ξ, a Fence mechanism is GSP between vectors with a stable pair

Definition

Let $b_i^* > \max_S \xi(i, S)$ for $i \in A$. $L = \{i \mid b_i \leq b_i^*\}$ for each b and U the set it yields.

Proof Sketch (cont.)

Induction on $|L|$ to prove that a stable pair exists; base trivial;

Induction step

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1. Assumption: no stable pair

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Induction on $|L|$ to prove that a stable pair exists; base trivial;

Induction step

1. Assumption: no stable pair
2. Let $T \subseteq U \setminus L$, s.t. $i \in T \Rightarrow b_i > \xi^*(i, L, U)$

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Induction on $|L|$ to prove that a stable pair exists; base trivial;

Induction step

1. Assumption: no stable pair
2. Let $T \subseteq U \setminus L$, s.t. $i \in T \Rightarrow b_i > \xi^*(i, L, U)$
3. L_i, U_i stable pair at (b_i^*, b_{-i}) $i \in T$

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4. $L \cup \{i\} \subseteq L_i \subseteq L \cup T$

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4. $L \cup \{i\} \subseteq L_i \subseteq L \cup T$
5. Fence Mon. Cond. 2. $\Rightarrow L \cup \{i\} \subset L_i$

Proof Sketch (cont.)

Induction on $|L|$ to prove that a stable pair exists; base trivial;

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4. $L \cup \{i\} \subseteq L_i \subseteq L \cup T$
5. Fence Mon. Cond. 2. $\Rightarrow L \cup \{i\} \subset L_i$
6. Let $j \in L_i \setminus (L \cup \{i\})$ $(j \in T)$

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Induction on $|L|$ to prove that a stable pair exists; base trivial;

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5. Fence Mon. Cond. 2. $\Rightarrow L \cup \{i\} \subset L_i$
6. Let $j \in L_i \setminus (L \cup \{i\})$ $(j \in T)$
7. Uniqueness and GSP $\Rightarrow L_j \subset L_i$

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6. Let $j \in L_i \setminus (L \cup \{i\})$ $(j \in T)$
7. Uniqueness and GSP $\Rightarrow L_j \subset L_i$
8. Contradiction!

Budget Balance and Complexity

Theorem

There is no general GSP mechanism with constant approximation ratio. That is there are cost function families, where every fence monotone ξ is at most $\frac{1}{x}$-budget balanced for any x.

Theorem

Finding the stable pair of an input is no harder than computing the outcome of a GSP mechanism given polynomial-time access to $\xi^*(\cdot,\cdot,\cdot)$.

Open Problems

- Prove lower bounds for interesting combinatorial problems
- Construct new GSP mechanisms with better budget balance
- Find the complexity of computing the outcome of a GSP mechanism
- Given a hardness result characterize GSP tractable mechanisms
- Characterize GSP mechanisms in other domains

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THANK YOU!

QUESTIONS?